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On the nonminimal vector coupling in the Duffin–Kemmer–Petiau theory and the confinement of massive bosons by a linear potential

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Abstract

The vector couplings in the Duffin–Kemmer–Petiau (DKP) theory have been revised. It is shown that minimal and nonminimal vector potentials behave differently under charge-conjugation and time-reversal transformations. In particular, it is shown that nonminimal vector potentials have been erroneously applied to the description of elastic meson–nucleus scatterings and that the space component of the nonminimal vector potential plays a crucial role for the confinement of bosons. The DKP equation with nonminimal vector linear potentials is mapped into the nonrelativistic harmonic oscillator problem and the behavior of the solutions for this sort of DKP oscillator is discussed in detail. Furthermore, the absence of Klein’s paradox and the localization of bosons in the presence of nonminimal vector interactions are discussed.

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1. Introduction

The first-order Duffin–Kemmer–Petiau (DKP) formalism [1, 2] describes spin-0 and spin-1 particles and has been used to analyze relativistic interactions of spin-0 and spin-1 hadrons with nuclei as an alternative to their conventional second-order Klein–Gordon and Proca counterparts. The onus of equivalence between the formalisms represented an objection to the DKP theory for a long time and it has only recently been shown that they yield the same results in the case of minimally coupled vector interactions, on the condition that one correctly interprets the components of the DKP spinor [3, 4]. However, the equivalence between the DKP and the Proca formalisms already has a precedent [5]. The equivalence does not hold if one considers partially conserved currents [6] and the DKP formalism proved to be better than the Klein–Gordon formalism in the analysis of the K_{l3} decays, the decay-rate ratio $\Gamma(\eta \rightarrow \gamma\gamma)/\Gamma(\pi^0 \rightarrow \gamma\gamma)$, and level shifts and widths

in pionic atoms [7]. Furthermore, the DKP formalism enjoys a richness of couplings which is not capable of being expressed in the Klein–Gordon and Proca theories. A number of different couplings in the DKP formalism, with scalar and vector couplings in the analogy with the Dirac phenomenology for proton–nucleus scattering, have been employed in the phenomenological treatment of the elastic meson–nucleus scattering at medium energies with a better agreement to the experimental data when compared to the Klein–Gordon and Proca-based formalisms [8–13]. On the other hand, the DKP theory has also experienced a renewed interest due to the discovery of a new conserved vector current [3, 14–20], whose positive-definite time component would be a candidate to a probability current, and as a bonus a hope for avoiding Klein’s paradox for bosons [20]. However, it has been shown that the proposed new current is a fiasco as a probability current [21]. An effort to disembarass the status of that new current was done [22] but in [23] it was shown to be indefensible. In [23] it also was shown that Klein’s paradox may exist in the DKP theory with minimally coupled vector interactions. The DKP theory has also experienced a renewal of life in the context of applications to quantum chromodynamics [24], covariant Hamiltonian dynamics [25], relativistic phase space [26], curved spacetime [27], causal approach [4, 28], superluminal tunneling [16], Bohm model [15, 17, 21], tunneling time [18], S-matrix [29], five-dimensional Galilean invariance [30], pseudoclassical mechanics [31], Bose–Einstein condensation [32], homogeneous magnetic field [33], Aharonov–Casher phase [34], Aharonov–Bohm potential [35], position-dependent mass and vector step potential [36], time-dependent mass and time-dependent vector fields [37], tensor DKP oscillator (tensor coupling with a linear potential) [38–45] and its thermodynamics properties [46], vector DKP oscillator (nonminimal vector coupling with a quadratic potential [39] and minimal plus nonminimal vector couplings with a linear potential [41]), sextic oscillator (tensor coupling with a linear plus a cubic potential) [47], vector step potential [20, 23, 48], vector Woods–Saxon potential [49], vector deformed Hulthen potential [50], vector square well [51], vector Coulomb potentials [43, 45, 51–53] and nonminimal vector step potentials [54].

The main purpose of the present paper is to report on the properties of the DKP theory with the nonminimal vector coupling interaction. Nonminimal vector potentials, added by other kinds of Lorentz structures, have already been used successfully in a phenomenological context for describing the scattering of mesons by nuclei [8, 9, 11, 13]. In this paper it is shown that charge-conjugation and time-reversal symmetries have some special features not displayed by minimal vector potentials, in particular the nonminimal vector potentials do not couple to the charge. It is also shown that nonminimal vector couplings have been used improperly in the phenomenological description of the elastic meson–nucleus scatterings [8, 9, 11, 13]. Furthermore, nonminimal vector potentials can be used as a model for confining bosons and that linear potentials lead to a sort of relativistic DKP oscillator.

This paper is organized as follows. In section 2 we present the general DKP equation, discuss conditions on the interactions which lead to a conserved current and effects of parity, charge-conjugation and time-reversal transformations on the vector Lorentz structures. Adopting a specific representation for the DKP matrices, we set up the one-dimensional equations for the components of the DKP spinor (IIA for the spin-0 sector and IIB for the spin-1 sector) in the presence of minimal and nonminimal vector interactions. We point out that the space component of the nonminimal vector potential cannot be absorbed into the spinor, as diffused in the literature. Beyond that, we show that the space component of the nonminimal vector potential could be irrelevant for the formation of bound states for the potentials vanishing at infinity but its presence is a *sine qua non* condition for confinement. In section 3 we focus on the nonminimal vector linear potentials and discuss the solutions of the vector DKP oscillator in detail. The relevance of the nonminimal vector potential for

the confinement of bosons is reinforced. An apparent paradox related to the localization of bosons in the presence of strong potentials is solved by introducing the concepts of effective mass and effective Compton wavelength. Finally, in section 4 we draw some conclusions.

2. The DKP equation and the vector couplings

The DKP equation for a free boson is given by [2] (with units in which $\hbar = c = 1$)

$$(i\beta^\mu \partial_\mu - m)\psi = 0, \quad (1)$$

where the matrices β^μ satisfy the algebra

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu} \beta^\lambda + g^{\lambda\nu} \beta^\mu \quad (2)$$

and the metric tensor is $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The algebra expressed by (2) generates a set of 126 independent matrices whose irreducible representations are a trivial representation, a five-dimensional representation describing the spin-0 particles and a ten-dimensional representation associated to spin-1 particles. The DKP spinor has an excess of components and the theory has to be supplemented by an equation which allows us to eliminate the redundant components. That constraint equation is obtained by multiplying the DKP equation by $1 - \beta^0 \beta^0$, namely

$$i\beta^j \beta^0 \beta^0 \partial_j \psi = m(1 - \beta^0 \beta^0)\psi, \quad j \text{ runs from 1 to 3.} \quad (3)$$

This constraint equation expresses three (four) components of the spinor by the other two (six) components and their space derivatives in the scalar (vector) sector so that the superfluous components disappear and there only the physical components of the DKP theory remain. The second-order Klein–Gordon and Proca equations are obtained when one selects the spin-0 and spin-1 sectors of the DKP theory.

A well-known conserved four-current is given by

$$J^\mu = \frac{1}{2} \bar{\psi} \beta^\mu \psi, \quad (4)$$

where the adjoint spinor $\bar{\psi}$ is given by

$$\bar{\psi} = \psi^\dagger \eta^0 \quad (5)$$

with

$$\eta^\mu = 2\beta^\mu \beta^\mu - g^{\mu\mu} \text{ (no summation)} \quad (6)$$

in such a way that $(\eta^0 \beta^\mu)^\dagger = \eta^0 \beta^\mu$ (the matrices β^μ are Hermitian with respect to η^0). The time component of this current is not positive definite but it may be interpreted as a charge density. The factor 1/2 multiplying $\bar{\psi} \beta^\mu \psi$, of no importance regarding the conservation law, is in order to hand over a charge density conformable to that one used in the Klein–Gordon theory and its nonrelativistic limit (see e.g. [55]). Then the normalization condition $\int d\tau J^0 = \pm 1$ can be expressed as

$$\int d\tau \bar{\psi} \beta^0 \psi = \pm 2, \quad (7)$$

where the plus (minus) sign must be used for a positive (negative) charge, and the expectation value of any observable \mathcal{O} may be given by

$$\langle \mathcal{O} \rangle = \frac{\int d\tau \bar{\psi} \beta^0 \mathcal{O} \psi}{\int d\tau \bar{\psi} \beta^0 \psi}, \quad (8)$$

where \mathcal{O} must be Hermitian with respect to β^0 , namely $(\beta^0 \mathcal{O})^\dagger = \beta^0 \mathcal{O}$, for ensuring that $\langle \mathcal{O} \rangle$ is a real quantity.

With the introduction of interactions, the DKP equation can be written as

$$(i\beta^\mu \partial_\mu - m - V)\psi = 0, \quad (9)$$

where the more general potential matrix V is written in terms of 25 (100) linearly independent matrices pertinent to the five (ten)-dimensional irreducible representation associated with the scalar (vector) sector. In the presence of interactions, J^μ satisfies the equation

$$\partial_\mu J^\mu + \frac{i}{2} \bar{\psi} (V - \eta^0 V^\dagger \eta^0) \psi = 0. \quad (10)$$

Thus, if V is Hermitian with respect to η^0 then the four-current will be conserved. The potential matrix V can be written in terms of well-defined Lorentz structures. For the spin-0 sector there are two scalar, two vector and two tensor terms [56], whereas for the spin-1 sector there are two scalar, two vector, a pseudoscalar, two pseudovector and eight tensor terms [57]. The tensor terms have been avoided in applications because they furnish noncausal effects [56, 57]. Considering only the vector terms, V is in the form

$$V = \beta^\mu A_\mu^{(1)} + i[P, \beta^\mu]A_\mu^{(2)}, \quad (11)$$

where P is a projection operator ($P^2 = P$ and $P^\dagger = P$) in such a way that $\bar{\psi} P \psi$ behaves as a scalar and $\bar{\psi} [P, \beta^\mu] \psi$ behaves like a vector. Note that the vector potential $A_\mu^{(1)}$ is minimally coupled but not $A_\mu^{(2)}$. One very important point to note is that this matrix potential leads to a conserved four-current but the same does not happen if instead of $i[P, \beta^\mu]$ one uses either $P\beta^\mu$ or $\beta^\mu P$, as in [8, 9, 11, 13, 39]. As a matter of fact, in [8] it is mentioned that $P\beta^\mu$ and $\beta^\mu P$ produce identical results.

If the terms in the potential V are time-independent one can write $\psi(\vec{r}, t) = \phi(\vec{r}) \exp(-iEt)$, where E is the energy of the boson, in such a way that the time-independent DKP equation becomes

$$[\beta^0(E - A_0^{(1)}) + i\beta^i(\partial_i + iA_i^{(1)}) - (m + i[P, \beta^\mu]A_\mu^{(2)})]\phi = 0. \quad (12)$$

In this case $J^\mu = \bar{\phi} \beta^\mu \phi / 2$ does not depend on time, so that the spinor ϕ describes a stationary state. Note that the time-independent DKP equation is invariant under a simultaneous shift of E and $A_0^{(1)}$, such as in the Schrödinger equation, but the invariance does not maintain regarding E and $A_0^{(2)}$. Equation (12) for the characteristic pair (E_k, ϕ_k) can be written as

$$(E_k - A_0^{(1)})\beta^0 \phi_k + i(\vec{\partial}_i + iA_i^{(1)})\beta^i \phi_k - m\phi_k - iA_\mu^{(2)}[P, \beta^\mu]\phi_k = 0 \quad (13)$$

and its adjoint form, by changing k by k' , as

$$(E_{k'} - A_0^{(1)})\bar{\phi}_{k'}\beta^0 - i\bar{\phi}_{k'}\beta^i\eta^0(\overleftarrow{\partial}_i - iA_i^{(1)}) - m\bar{\phi}_{k'}\eta^0 + iA_\mu^{(2)}\bar{\phi}_{k'}\eta^0[P, \beta^\mu]^\dagger = 0. \quad (14)$$

By multiplying (13) from the left by $\bar{\phi}_{k'}$ and (14) from the right by $\eta^0 \phi_k$ leads to

$$(E_k - A_0^{(1)})\bar{\phi}_{k'}\beta^0 \phi_k + i\bar{\phi}_{k'}\beta^i(\vec{\partial}_i + iA_i^{(1)})\phi_k - m\bar{\phi}_{k'}\phi_k - iA_\mu^{(2)}\bar{\phi}_{k'}[P, \beta^\mu]\phi_k = 0 \quad (15)$$

and

$$(E_{k'} - A_0^{(1)})\bar{\phi}_{k'}\beta^0 \phi_k - i\bar{\phi}_{k'}\beta^i(\overleftarrow{\partial}_i - iA_i^{(1)})\phi_k - m\bar{\phi}_{k'}\phi_k + iA_\mu^{(2)}\bar{\phi}_{k'}\eta^0[P, \beta^\mu]^\dagger \eta^0 \phi_k = 0, \quad (16)$$

respectively. Subtracting (16) from (15) and considering that the spinors fit boundary conditions such that

$$\int d\tau \partial_i (\bar{\phi}_k \beta^i \phi_{k'}) = 0, \quad (17)$$

one gets

$$(E_k - E_{k'}) \int d\tau \bar{\phi}_k \beta^0 \phi_{k'} = 0. \quad (18)$$

Equation (18) is an orthogonality statement applying to the DKP equation. Any two stationary states with distinct energies and subject to suitable boundary conditions are orthogonal in the sense that

$$\int d\tau \bar{\phi}_k \beta^0 \phi_{k'} = 0, \quad \text{for } E_k \neq E_{k'}. \quad (19)$$

In addition, in view of (7) the spinors ϕ_k and $\phi_{k'}$ are said to be orthonormal if

$$\int d\tau \bar{\phi}_k \beta^0 \phi_{k'} = \pm 2\delta_{E_k E_{k'}}. \quad (20)$$

The DKP equation is invariant under the parity operation, i.e. when $\vec{r} \rightarrow -\vec{r}$, if $A_i^{(1)}$ and $A_i^{(2)}$ change sign, whereas $A_0^{(1)}$ and $A_0^{(2)}$ remain the same. This is because the parity operator is $\mathcal{P} = \exp(i\delta_P)P_0\eta^0$, where δ_P is a constant phase and P_0 changes \vec{r} into $-\vec{r}$. Because this unitary operator anticommutes with β^i and $[P, \beta^i]$, they change sign under a parity transformation, whereas β^0 and $[P, \beta^0]$, which commute with η^0 , remain the same. Since $\delta_P = 0$ or $\delta_P = \pi$, the spinor components have definite parities. The charge-conjugation operation changes the sign of the minimal interaction potential, i.e. changes the sign of $A_\mu^{(1)}$. This can be accomplished by the transformation $\psi \rightarrow \psi_c = C\psi = CK\psi$, where K denotes the complex conjugation and C is a unitary matrix such that $C\beta^\mu = -\beta^\mu C$. The matrix that satisfies this relation is $C = \exp(i\delta_C)\eta^0\eta^1$. The phase factor $\exp(i\delta_C)$ is equal to ± 1 ; thus $E \rightarrow -E$. Note also that $J^\mu \rightarrow -J^\mu$, as should be expected for a charge current. Meanwhile C anticommutes with $[P, \beta^\mu]$ and the charge-conjugation operation entails no change on $A_\mu^{(2)}$. By the same token, it can be shown that $A_\mu^{(1)}$ and $A_\mu^{(2)}$ have opposite behavior under the time-reversal transformation in such a way that both sorts of vector potentials change sign under \mathcal{PCT} . The invariance of the nonminimal vector potential under charge conjugation means that it does not couple to the charge of the boson. In other words, $A_\mu^{(2)}$ does not distinguish particles from antiparticles. Hence, whether one considers spin-0 or spin-1 bosons, this sort of interaction cannot exhibit Klein's paradox.

2.1. Scalar sector

For the case of spin-0, we use the representation for the β^μ matrices given by [52]

$$\beta^0 = \begin{pmatrix} \theta & \bar{\mathbf{0}} \\ \bar{\mathbf{0}}^T & \mathbf{0} \end{pmatrix}, \quad \beta^i = \begin{pmatrix} \tilde{\mathbf{0}} & \rho_i \\ -\rho_i^T & \mathbf{0} \end{pmatrix}, \quad i = 1, 2, 3, \quad (21)$$

where

$$\begin{aligned} \theta &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \rho_1 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \rho_2 &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \rho_3 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (22)$$

where $\bar{\mathbf{0}}$, $\tilde{\mathbf{0}}$ and $\mathbf{0}$ are 2×3 , 2×2 and 3×3 zero matrices, respectively, while the superscript T designates matrix transposition. Here the projection operator can be written as [56]

$$P = \frac{1}{3}(\beta^\mu \beta_\mu - 1) = \text{diag}(1, 0, 0, 0, 0). \quad (23)$$

In this case P picks out the first component of the DKP spinor. The five-component spinor can be written as $\psi^T = (\psi_1, \dots, \psi_5)$ in such a way that the DKP equation for a boson constrained to move along the X -axis decomposes into

$$\begin{aligned} (D_0^{(-)} D_0^{(+)} - D_1^{(-)} D_1^{(+)} + m^2)\psi_1 &= 0 \\ D_0^{(+)}\psi_1 &= -im\psi_2, & D_1^{(+)}\psi_1 &= -im\psi_3 \\ \psi_4 &= \psi_5 = 0, \end{aligned} \quad (24)$$

where

$$D_\mu^{(\pm)} = \partial_\mu + iA_\mu^{(1)} \pm A_\mu^{(2)}. \quad (25)$$

Furthermore,

$$\begin{aligned} J^0 &= \text{Re}(\psi_2^* \psi_1) = -\frac{1}{m} \text{Im}(\psi_1^* D_0^{(+)} \psi_1) \\ J^1 &= -\text{Re}(\psi_3^* \psi_1) = \frac{1}{m} \text{Im}(\psi_1^* D_1^{(+)} \psi_1) \\ J^2 &= J^3 = 0. \end{aligned} \quad (26)$$

Note that, in the absence of the nonminimal potential, the first line of (24) reduces to the Klein–Gordon equation and that ψ_3 , ψ_4 and ψ_5 are the superfluous components of the DKP spinor (the reason that $\psi_4 = \psi_5 = 0$ is because of the one-dimensional movement).

In the time-independent case, one has

$$\begin{aligned} \left(\frac{d^2}{dx^2} + 2iA_1^{(1)} \frac{d}{dx} + k^2 \right) \phi_1 &= 0 \\ \phi_2 &= \frac{1}{m} (E - A_0^{(1)} + iA_0^{(2)}) \phi_1 \\ \phi_3 &= \frac{i}{m} \left(\frac{d}{dx} + iA_1^{(1)} + A_1^{(2)} \right) \phi_1, \end{aligned} \quad (27)$$

where

$$k^2 = (E - A_0^{(1)})^2 - m^2 - (A_1^{(1)})^2 + i \frac{dA_1^{(1)}}{dx} + (A_0^{(2)})^2 - (A_1^{(2)})^2 + \frac{dA_1^{(2)}}{dx}. \quad (28)$$

Meanwhile,

$$J^0 = \frac{E - A_0^{(1)}}{m} |\phi_1|^2, \quad J^1 = \frac{1}{m} \left[A_1^{(1)} |\phi_1|^2 + \text{Im} \left(\phi_1^* \frac{d\phi_1}{dx} \right) \right]. \quad (29)$$

It is worthwhile to note that J^0 becomes negative in regions of space where $E < A_0^{(1)}$ (a circumstance associated with Klein’s paradox) and that $A_\mu^{(2)}$ does not intervene explicitly in the current. The orthonormalization formula (20) becomes

$$\int_{-\infty}^{+\infty} dx \frac{\frac{E_k + E_{k'}}{2} - A_0^{(1)}}{m} \phi_{1k}^* \phi_{1k'} = \pm \delta_{E_k E_{k'}} \quad (30)$$

regardless $A_1^{(1)}$ and $A_\mu^{(2)}$. Equation (30) is in agreement with the orthonormalization formula for the Klein–Gordon theory in the presence of a minimally coupled potential [55]. This is not surprising, because, after all, both DKP equation and Klein–Gordon equation are equivalent under minimal coupling.

The form $\partial_1 + iA_1^{(1)}$ in equation (24) suggests that the space component of the minimal vector potential can be gauged away by defining a new spinor

$$\tilde{\psi} = \exp(i\Lambda) \psi, \quad A_1^{(1)} = \partial_1 \Lambda, \quad (31)$$

even if $A_1^{(1)}$ is time dependent. Without any question

$$(\partial_1 + iA_1^{(1)}) \psi = \exp(-i\Lambda) \partial_1 \tilde{\psi} \quad (32)$$

in such a way that $\tilde{\psi}$ satisfies the DKP equation without $A_1^{(1)}$. In [8] and [9] the term involving $A_1^{(2)}$ was explicitly absorbed into the wavefunction. Nevertheless, it seems that there is no chance to dissociate from this term. As a matter of fact, we will show that the space

component of the nonminimal vector potential plays a peremptory role for confining bosons. The possibility for ruling out $A_1^{(1)}$ but not $A_1^{(2)}$ is reinforced by the observation that the first derivative of a second-order differential equation, such as the term containing $A_1^{(1)}$ in the first line of equation (27), is a well-known trick in mathematics.

It is noticeable that if $|A_\mu^{(2)}| \rightarrow \infty$ as $|x| \rightarrow \pm\infty$, confining solutions for a pure nonminimal vector potential will be possible on the condition that the space component of $A_\mu^{(2)}$ is stronger, or has a dominant asymptotic behavior, than its time component. Otherwise, nothing but continuum states will be possible. In this last circumstance, a boson can tunnel into the classically forbidden region, an unexpected result in nonrelativistic mechanics and by no means related to Klein's paradox. On the other hand, for a pure nonminimal vector potential going to zero at infinity, a necessary condition for the existence of bound-state solutions (with $|E| < m$) is that

$$(A_0^{(2)})^2 - (A_1^{(2)})^2 + \frac{dA_1^{(2)}}{dx} > 0 \quad (33)$$

at any arbitrary point on the X -axis. In this case, it is the time component of the nonminimal vector potential that plays a leading role in establishing bound states.

2.2. Vector sector

For the case of spin 1, the β^μ matrices are [51]

$$\beta^0 = \begin{pmatrix} 0 & \bar{0} & \bar{0} & \bar{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \bar{0}^T & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & \bar{0} & e_i & \bar{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{0} & -is_i \\ -e_i^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \bar{0}^T & -is_i & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (34)$$

where s_i are the 3×3 spin-1 matrices $(s_i)_{jk} = -i\varepsilon_{ijk}$, e_i are the 1×3 matrices $(e_i)_{1j} = \delta_{ij}$ and $\bar{0} = (0 \ 0 \ 0)$, while \mathbf{I} and $\mathbf{0}$ designate the 3×3 unit and zero matrices, respectively. In this representation,

$$P = \beta^\mu \beta_\mu - 2 = \text{diag}(1, 1, 1, 1, 0, 0, 0, 0, 0, 0), \quad (35)$$

i.e. P projects out the four upper components of the DKP spinor. With the spinor written as $\psi^T = (\psi_1, \dots, \psi_{10})$, and partitioned as

$$\begin{aligned} \psi_I^{(+)} &= \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, & \psi_I^{(-)} &= \psi_5 \\ \psi_{II}^{(+)} &= \begin{pmatrix} \psi_6 \\ \psi_7 \end{pmatrix}, & \psi_{II}^{(-)} &= \psi_2 \\ \psi_{III}^{(+)} &= \begin{pmatrix} \psi_{10} \\ -\psi_9 \end{pmatrix}, & \psi_{III}^{(-)} &= \psi_1 \end{aligned} \quad (36)$$

the one-dimensional DKP equation can be expressed in the compact form

$$\begin{aligned} (D_0^{(\mp)} D_0^{(\pm)} - D_1^{(\mp)} D_1^{(\pm)} + m^2) \psi_I^{(\pm)} &= 0 \\ D_0^{(\pm)} \psi_I^{(\pm)} &= -im \psi_{II}^{(\pm)}, & D_1^{(\pm)} \psi_I^{(\pm)} &= -im \psi_{III}^{(\pm)} \\ \psi_8 &= 0, \end{aligned} \quad (37)$$

where $D_\mu^{(\pm)}$ is again given by (25). In addition, expressed in terms of (36) the current can be written as

$$\begin{aligned} J^0 &= \text{Re}(\psi_{II}^{(+)\dagger} \psi_I^{(+)} + \psi_{II}^{(-)\dagger} \psi_I^{(-)}) = -\frac{1}{m} \text{Im}(\psi_I^{(+)\dagger} D_0^{(+)} \psi_I^{(+)} + \psi_I^{(-)\dagger} D_0^{(-)} \psi_I^{(-)}) \\ J^1 &= -\text{Re}(\psi_{III}^{(+)\dagger} \psi_I^{(+)} + \psi_{III}^{(-)\dagger} \psi_I^{(-)}) = \frac{1}{m} \text{Im}(\psi_I^{(+)\dagger} D_1^{(+)} \psi_I^{(+)} + \psi_I^{(-)\dagger} D_1^{(-)} \psi_I^{(-)}) \\ J^2 &= J^3 = 0. \end{aligned} \quad (38)$$

Note that the third line plus the second equation in the middle line of (37) are the constraint equations which allow one to eliminate the superfluous components (ψ_1, ψ_8, ψ_9 and ψ_{10}) of the DKP spinor. The component $\psi_8 = 0$ because the movement is restricted to the X -axis.

Meanwhile the time-independent DKP equation decomposes into

$$\begin{aligned} \left(\frac{d^2}{dx^2} + 2iA_1^{(1)} \frac{d}{dx} + k_\pm^2 \right) \phi_I^{(\pm)} &= 0 \\ \phi_{II}^{(\pm)} &= \frac{1}{m} (E - A_0^{(1)} \pm iA_0^{(2)}) \phi_I^{(\pm)} \\ \phi_{III}^{(\pm)} &= \frac{i}{m} \left(\frac{d}{dx} + iA_1^{(1)} \pm A_1^{(2)} \right) \phi_I^{(\pm)}, \end{aligned} \quad (39)$$

where

$$k_\pm^2 = (E - A_0^{(1)})^2 - m^2 - (A_1^{(1)})^2 + i \frac{dA_1^{(1)}}{dx} + (A_0^{(2)})^2 - (A_1^{(2)})^2 \pm \frac{dA_1^{(2)}}{dx}. \quad (40)$$

Now the components of the four-current are

$$\begin{aligned} J^0 &= \frac{E - A_0^{(1)}}{m} (|\phi_I^{(+)}|^2 + |\phi_I^{(-)}|^2) \\ J^1 &= \frac{1}{m} \left[A_1^{(1)} (|\phi_I^{(+)}|^2 + |\phi_I^{(-)}|^2) + \text{Im} \left(\phi_I^{(+)\dagger} \frac{d\phi_I^{(+)}}{dx} + \phi_I^{(-)\dagger} \frac{d\phi_I^{(-)}}{dx} \right) \right] \end{aligned} \quad (41)$$

and the orthonormalization expression (20) takes the form

$$\int_{-\infty}^{+\infty} dx \frac{\frac{E_k + E_{k'}}{2} - A_0^{(1)}}{m} (\phi_{Ik}^{(+)\dagger} \phi_{Ik'}^{(+)} + \phi_{Ik}^{(-)\dagger} \phi_{Ik'}^{(-)}) = \pm \delta_{E_k E_{k'}}. \quad (42)$$

Just as for scalar bosons, $J^0 < 0$ for $E < A_0^{(1)}$ and $A_\mu^{(2)}$ does not appear in the current. Similarly, $A_1^{(1)}$ and $A_\mu^{(2)}$ do not manifest explicitly in the orthonormalization formula.

From (39) to (40), one sees that the solution for the spin-1 sector consists in searching solutions for two Klein–Gordon-like equations, owing to the term $dA_1^{(2)}/dx$ in (40). It should not be forgotten, though, that the equations for $\phi_I^{(+)}$ and $\phi_I^{(-)}$ are not indeed independent because E appears in both equations. Evidently, matching a common value for the energy might compromise the existence of solutions for spin-1 bosons when compared to the solutions for spin-0 bosons with the very same potentials. This amounts to say that the solutions for the spin-1 sector of the DKP theory, if they really exist, can be obtained from a restrict class of solutions of the spin-0 sector. This limitation on the possible solutions for spin-1 bosons as compared for spin-0 bosons should not be a surprise if one remembers that, in the absence of any interaction, all the components of the free Proca equation obey a free Klein–Gordon equation but with an additional constraint on the components of the Proca field.

3. The nonminimal vector linear potential

Having set up the spin-0 and spin-1 equations for vector interactions, we are now in a position to use the machinery developed above in order to solve the DKP equation with specific forms for nonminimal interactions. Let us consider pure nonminimal vector linear potentials in the form

$$A_0^{(2)} = m^2\omega_0|x|, \quad A_1^{(2)} = m^2\omega_1x, \quad (43)$$

where ω_0 and ω_1 are dimensionless quantities. Our problem is to solve (27) and (39) for ϕ and to determine the allowed energies. Although the absolute value of x in $A_0^{(2)}$ is irrelevant in the effective equations for ϕ_1 (in the scalar sector) and $\phi_I^{(\pm)}$ (in the vector sector), it is there for ensuring the covariance of the DKP theory under parity. It follows that the DKP spinor will have a definite parity and A^μ and J^μ will be genuine four-vectors.

3.1. Scalar sector

For the spin-0 sector of the DKP theory one finds that ϕ_1 obeys the second-order differential equation

$$\frac{d^2\phi_1}{dx^2} + (\varepsilon^2 - m^4\Omega^2x^2)\phi_1 = 0, \quad (44)$$

where

$$\varepsilon^2 = E^2 - m^2 + m^2\omega_1, \quad \Omega^2 = \omega_1^2 - \omega_0^2. \quad (45)$$

The solution for (44), with $\varepsilon^2 > 0$ and $\Omega^2 > 0$, is precisely the well-known solution of the Schrödinger equation for the nonrelativistic harmonic oscillator (see, e.g. [58])

$$\varepsilon_n^2 = (2n + 1)m^2|\Omega| \quad (46)$$

$$(\phi_1)_n = N_n H_n(m\sqrt{|\Omega|x}) \exp\left(-\frac{m^2|\Omega|x^2}{2}\right), \quad (47)$$

where $n = 0, 1, 2, \dots, N_n$ is a normalization constant, and $H_n(\zeta)$ is an n th degree Hermite polynomial in ζ . Note that the condition $\Omega^2 > 0$ requires that $|\omega_1| > |\omega_0|$, meaning that the space component of the potential must be stronger than its time component in order to the effective potential be a true confining potential. Nevertheless, there is no requirement on the signs of ω_1 and ω_0 . From (46) one obtains the discrete set of DKP energies (symmetrical about $E = 0$ as it should be since $A_\mu^{(2)}$ does not distinguish particles from antiparticles) $E_n = \pm|E_n|$, where

$$|E_n| = m\sqrt{1 - \omega_1 + (2n + 1)|\Omega|} \quad (48)$$

irrespective to the sign of ω_0 . In general, $|E_n|$ is higher for $\omega_1 < 0$ than for $\omega_1 > 0$. It increases with the quantum number and it is a monotonically decreasing function of ω_0 . In order to ensure the reality of the spectrum, the coupling constants ω_0 and ω_1 satisfy the additional constraint

$$(2n + 1)\sqrt{\omega_1^2 - \omega_0^2} \geq \omega_1 - 1. \quad (49)$$

If one squares (49) the resulting inequality is in general a quadratic algebraic inequality in ω_1 (or ω_0), which can be solved analytically. The price paid is that some spurious solutions can appear in this process, although, of course, these can be eliminated by checking whether they satisfy the original inequality. A more instructive procedure is to follow a graphical method,

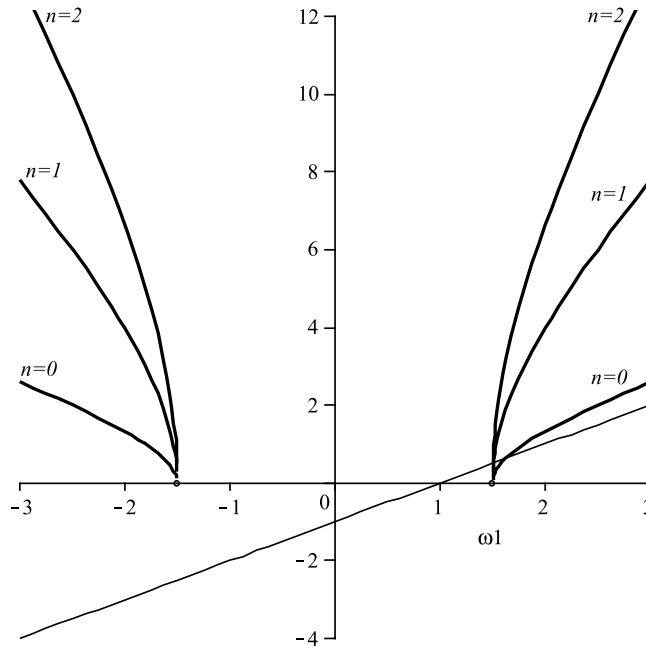


Figure 1. Graphical solution of (49) for $|\omega_0| = 1.5$ for the three lowest quantum numbers. Heavy lines for $f_H(\omega_1)$ and light line for $f_S(\omega_1)$.

by which one seeks the regions of the functions of ω_1 in (49): a hyperbole on the left-hand side,

$$f_H(\omega_1) = (2n + 1)\sqrt{\omega_1^2 - \omega_0^2}, \tag{50}$$

and a straight line on the right-hand side,

$$f_S(\omega_1) = \omega_1 - 1, \tag{51}$$

where $f_H(\omega_1)$ is a nonnegative function having two symmetric branches and for $|\omega_1| \gg |\omega_0|$ it approximates the function $(2n + 1)|\omega_1|$. Figure 1 present results for the three first quantum numbers with $|\omega_0| > 1$. For $\omega_1 > |\omega_0|$, this figure shows clearly that $f_H \geq f_S$ only for some $\omega_1 \geq (\tilde{\omega}_1)_n > |\omega_0|$, although $f_H > f_S$ for all $\omega_1 < -|\omega_0|$. The intersection points of f_H and f_S , for $|\omega_0| > 1$, correspond to $|E_n| = 0$. Figure 1 also allows one to conclude that $|E_n| > 0$ for $|\omega_0| < 1$. Note that there is a high density of states (number of states in a fixed range of energy) corresponding to an infinite set of quasi-degenerate solutions in the neighborhood of $|\omega_1| = |\omega_0|$.

In the weak-coupling limit, $\omega_1 \ll 1$ and $|\Omega| \ll 1$, $|E_n| \simeq m$ for small quantum numbers and (48) becomes

$$|E_n| \simeq m \left[1 - \frac{\omega_1}{2} + \left(n + \frac{1}{2} \right) |\Omega| \right]. \tag{52}$$

This equally spaced energy spectrum is a sort of nonrelativistic limit. Therefore, it can be said that the linear potentials given by (43) describe a genuine nonminimal vector DKP oscillator. Nevertheless, the Lorentz structure of the potentials plays no role in a nonrelativistic scheme, because one has to use the Schrödinger equation with the potential $A_0^{(2)} + A_1^{(2)}$. Despite the

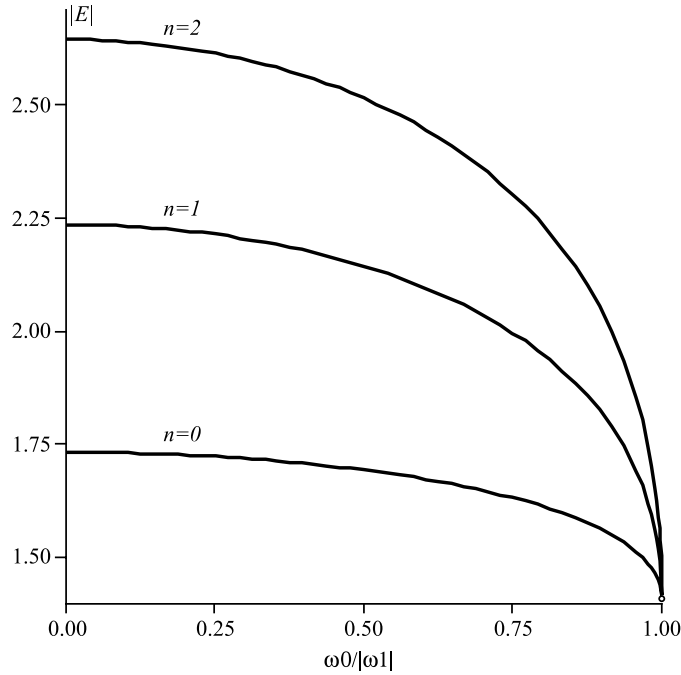


Figure 2. Positive spectrum of spin-0 bosons for the three lowest quantum numbers as a function of $\omega_0/|\omega_1|$, for $\omega_1 = -1$ ($m = 1$).

effective harmonic oscillator potential appearing in (44), the linear potentials given by (43) do not furnish bound-state solutions in the Schrödinger equation because the sum $A_0^{(2)} + A_1^{(2)}$ with $A_1^{(2)} \neq 0$ is unbounded from below.

On the other hand, for $|\omega_1| \gg |\omega_0|$ one has that

$$|E_n| \simeq m\sqrt{1 - \omega_1 + (2n + 1)|\omega_1|}, \tag{53}$$

so that $|E_n| > m$ for $\omega_1 < 0$. Concerning $\omega_1 > 0$, as far as ω_1 increases, the spectrum moves toward $E = 0$, except for $\omega_0 = 0$ which maintains $|E_n| \geq m$ (the spectrum acquiesces $|E_0| = m$ in this limit case).

Figures 2, 3 and 4 illustrate the spectrum in terms of $\omega_0/|\omega_1|$ for three different values of ω_1 . For $\omega_1 < 1$ there is a spectral gap given by

$$2m\sqrt{1 - \omega_1 + (2n + 1)|\omega_1|} \tag{54}$$

and there are infinitely many energy levels above m where, in the absence of interaction, there was the continuum. As far as ω_1 increases, the spectrum moves toward $E = 0$, except for $\omega_0 = 0$. The gap tends to vanish as ω_1 becomes close to 1 (for $\omega_0 \neq 0$) and so the positive- and negative-energy levels tend to be very close to each other. Figures 5 and 6 illustrate the spectrum in terms of ω_1/ω_0 for two different values of ω_0 .

The charge density

$$J^0 = \frac{E}{m} |\phi_1|^2 \tag{55}$$

dictates that ϕ_1 must be normalized as

$$\frac{|E|}{m} \int_{-\infty}^{+\infty} dx |\phi_1|^2 = 1. \tag{56}$$

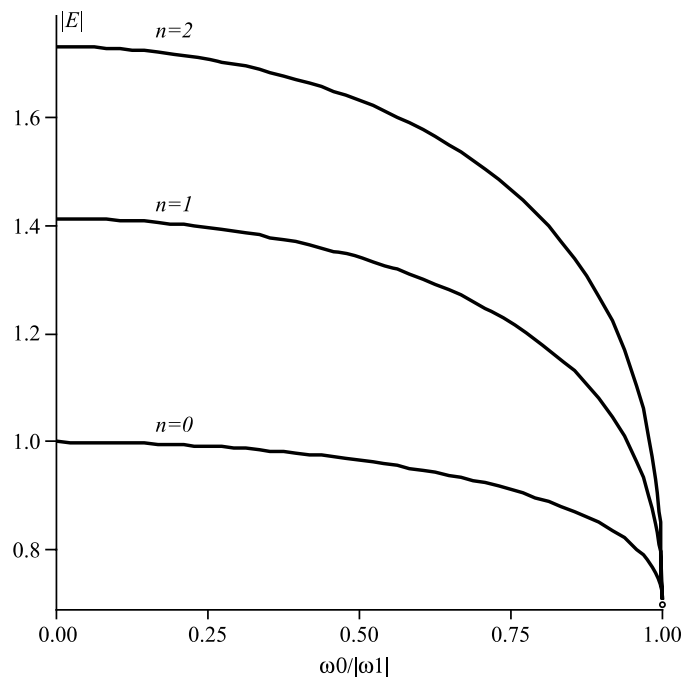


Figure 3. The same as figure 2, for $\omega_1 = 0.5$.

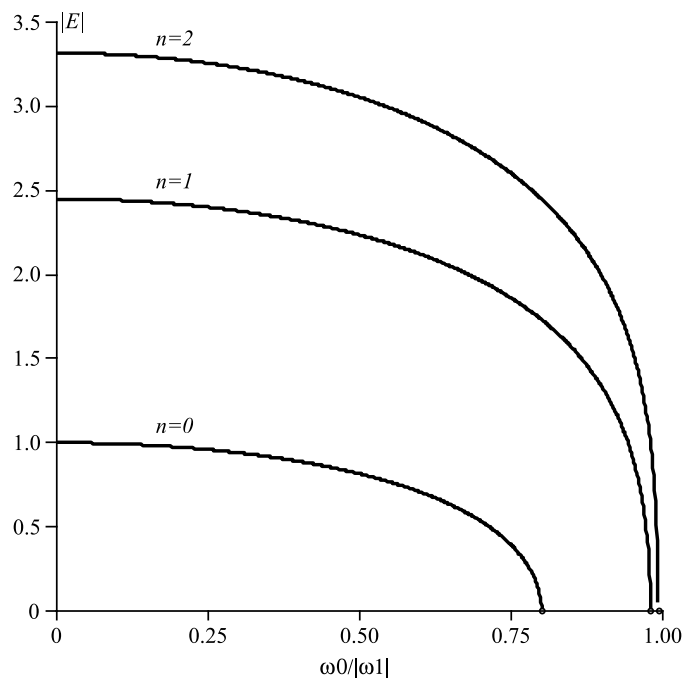


Figure 4. The same as figure 2, for $\omega_1 = 2.5$.

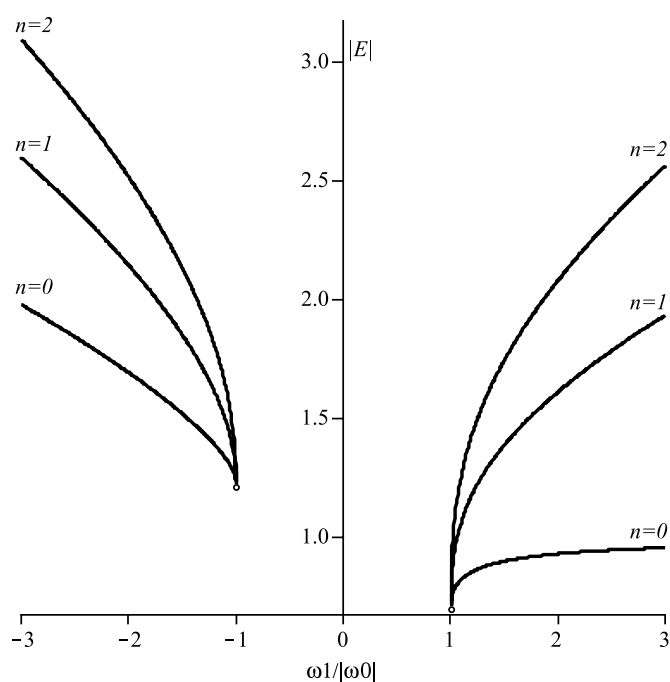


Figure 5. Positive spectrum of spin-0 bosons for the three lowest quantum numbers as a function ω_1/ω_0 , for $\omega_0 = 0.5$ ($m = 1$).

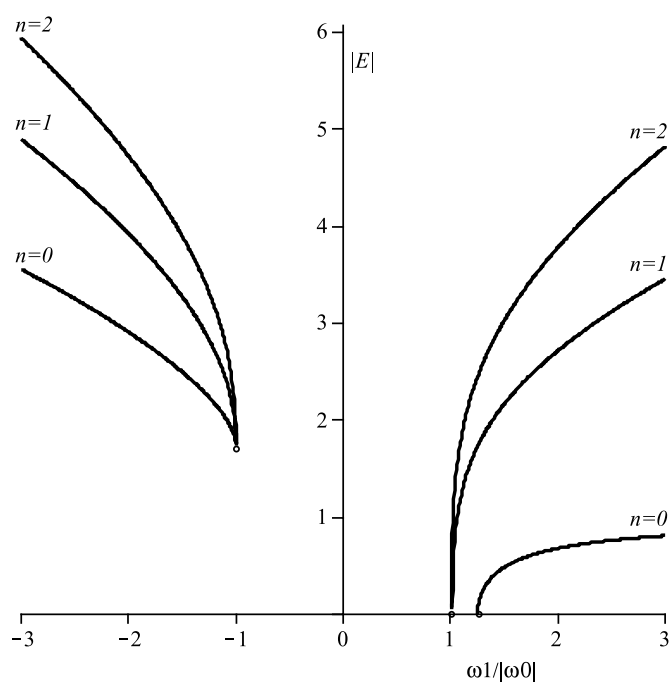


Figure 6. The same as figure 5, for $\omega_0 = 2$.

Using the property [58]

$$\int_{-\infty}^{+\infty} d\zeta H_n^2(\zeta) \exp(-\zeta^2) = 2^n n! \sqrt{\pi}, \quad (57)$$

one finds that the normalization constant can be chosen to be

$$N_n = \left(\frac{m|\Omega|}{\pi} \right)^{1/4} \sqrt{\frac{m}{2^n n! |E_n|}}, \quad \text{for } E_n \neq 0. \quad (58)$$

Thus, for $E_n \neq 0$, one has

$$J_n^0(x) = \frac{\text{sign}(E_n)}{2^n n!} \sqrt{\frac{m|\Omega|}{\pi}} H_n^2(m\sqrt{|\Omega|x}) \exp(-m^2|\Omega|x^2). \quad (59)$$

Then, using (8), the quantity $(\Delta x_n)^2 = \langle x^2 \rangle_n - \langle x \rangle_n^2$ can be written as

$$(\Delta x_n)^2 = \int_{-\infty}^{+\infty} dx |J_n^0(x)| x^2 - \left(\int_{-\infty}^{+\infty} dx |J_n^0(x)| x \right)^2.$$

Now it is a simple matter to write down the uncertainty in the position

$$\Delta x_n = \sqrt{\frac{n + 1/2}{m^2|\Omega|}}. \quad (60)$$

If Δx_n shrinks then Δp_n (uncertainty in the momentum) will swell, in consonance with the Heisenberg uncertainty principle. Nevertheless, the maximum uncertainty in the momentum is given by m requiring that is impossible to localize a boson in a region of space less than half of its Compton wavelength (see, for example, [59, 60]). Nevertheless, if one defines an effective mass as $m_{\text{eff}} = m\sqrt{|\Omega|}$ and an effective Compton wavelength as $\lambda_{\text{eff}} = 1/m_{\text{eff}}$, one will find that $\Delta x_n = \lambda_{\text{eff}} \sqrt{n + 1/2}$. It follows that the high localization of bosons, related to high values of $|\Omega|$ ($|\omega_1| \gg |\omega_0|$), never menaces the single-particle interpretation of the DKP theory. For $|\Omega| \simeq 0$ ($|\omega_1| \simeq |\omega_0|$), one has that $m_{\text{eff}} \simeq 0$ and the quasi-degenerate solutions mentioned above are related to very delocalized states. As for the behavior in the neighborhood of $E_n = 0$ one should note that, despite J_n^0 and Δx_n being independent of E_n , the DKP spinor is not defined for $|E_n| = 0$. Thus, $E_n = 0$ must be ruled out of the theory. Although positive- and negative-energy levels do not touch, they can be very close to each other for moderately strong coupling constants without any danger of reaching the conditions for Klein's paradox.

3.2. Vector sector

As for the spin-1 sector, proceeding as before, one finds that $\phi_I^{(\pm)}$ obeys the equation

$$\frac{d^2 \phi_I^{(\pm)}}{dx^2} + (\varepsilon_{\pm}^2 - m^4 \Omega^2 x^2) \phi_I^{(\pm)} = 0, \quad (61)$$

where Ω^2 is defined as in (45) and

$$\varepsilon_{\pm}^2 = E^2 - m^2 \pm m^2 \omega_1. \quad (62)$$

For bound states, to which we shall devote our attention, we must require $\varepsilon_{\pm}^2 > 0$ and $\Omega^2 > 0$, as before. Thus, the solution is expressed as

$$\varepsilon_{n_{\pm}}^2 = (2n_{\pm} + 1) m^2 |\Omega| \quad (63)$$

$$(\phi_I^{(\pm)})_{n_{\pm}} = N_{n_{\pm}} H_{n_{\pm}}(m\sqrt{|\Omega|x}) \exp\left(-\frac{m^2|\Omega|x^2}{2}\right), \quad (64)$$

where $n_{\pm} = 0, 1, 2, \dots$, $N_{n_{\pm}}$ is a normalization constant and $N_{n_{\pm}} = (N_3, N_4)^T$ is a column matrix whose elements are normalization constants related to the solutions for ϕ_3 and ϕ_4 . Hence, the necessary conditions for binding spin-1 bosons subject to linear potentials have been put forward. The formal analytical solutions have been obtained and it has been revealed that the solutions related to the spinor $\phi_I^{(+)}$ are formally the same as those ones for spin-0 bosons. Now we move on to match a common energy to the spin-1 boson problem. The matching condition requires that the quantum numbers n_+ and n_- must satisfy the relation

$$n_+ - n_- = \frac{1}{\sqrt{1 - \left(\frac{\omega_0}{\omega_1}\right)^2}} \frac{\omega_1}{|\omega_1|}. \quad (65)$$

This constraint on the nodal structure of $\phi_I^{(+)}$ and $\phi_I^{(-)}$ dictates that acceptable solutions only occur for a countable number of possibilities for $|\omega_1|/|\omega_0|$, namely

$$\frac{|\omega_0|}{|\omega_1|} = \sqrt{1 - \frac{1}{(n_+ - n_-)^2}}. \quad (66)$$

4. Conclusions

We showed that minimal and nonminimal vector interactions behave differently under charge-conjugation and time-reversal transformations. Although Klein's paradox cannot be treated as unworthy of regard in the DKP theory with minimally coupled vector interactions, it never makes its appearance in the case of nonminimal vector interactions because they do not couple to the charge.

In the case of a pure nonminimal vector coupling, both particle and particle energy levels are members of the spectrum, and the particle and antiparticle spectra are symmetrical about $E = 0$. If the interaction potential is attractive (repulsive) for bosons it will also be attractive (repulsive) for antibosons. However, there is no crossing of levels because possible states in the strong field regime with $E = 0$ are in fact unnormalizable. These facts imply that there is no channel for spontaneous boson–antiboson creation and for that reason the single-particle interpretation of the DKP equation is ensured. The charge conjugation operation allows us to migrate from the spectrum of particles to the spectrum of antiparticles and vice versa just by changing the sign of E . This change induces no change in the nodal structure of the components of the DKP spinor and so the nodal structure of the four-current is preserved.

In view of recent developments on the construction of a positive-definite inner product for the Klein–Gordon theory [61], we acknowledge that we took a very conservative stance when considering a current that cannot be related to a probability current. The interesting possibility of a probability current in the DKP theory, constructed from the energy–momentum tensor, launched in [14, 15], though, received a severe criticism in [21] and [23]. It will then be challenging to construct a probability current in the DKP theory from a relativistically invariant positive-definite inner product. Notwithstanding, the conserved charge current plus the charge conjugation operation are enough to infer about the absence of Klein's paradox under nonminimal vector interactions, or its possible presence under minimal vector interactions.

We showed that nonminimal vector couplings have been used improperly in the phenomenological description of elastic meson–nucleus scatterings potential by observing that the four-current is not conserved when one uses either the matrix $P\beta^\mu$ or $\beta^\mu P$, even though the linear forms constructed from those matrices behave as true Lorentz vectors. We also pointed out that the space component of the nonminimal vector potential cannot be absorbed into the spinor. Beyond that, we showed that the space component of the nonminimal

vector potential could be irrelevant for the formation of bound states for potentials vanishing at infinity but its presence is an essential ingredient for confinement.

For the one-dimensional problem, the DKP equation with nonminimal vector potentials was mapped into a Sturm–Liouville problem in such a way that the solution for linear potentials could be found by solving a Schrödinger-like problem for the nonrelativistic harmonic oscillator. The behavior of the solutions for this sort of DKP oscillator was discussed in detail. That model reinforced the absence of Klein’s paradox. Furthermore, due to the fact that there is no room for the boson–antiboson production, a boson embedded in this sort of background acquires an effective mass which permits that it can be strictly localized. We also showed that the DKP oscillator for vector bosons is conditionally solvable.

In addition we provided a better understanding of the DKP theory with a coupling full of phenomenological relevance since it has not yet been well explored in the literature, it was conceived as an exactly solvable vector model relating to the confinement of bosons.

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